An axiomatic system for STIT imagination logic

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Abstract. We formulate a Hilbert-style axiomatic system for STIT logic of imagination recently proposed by H. Wansing in [2] and prove its completeness by the method of canonical models.

Keywords: STIT logic, logic of imagination, canonical models, completeness, axiomatization

We assume a propositional language with a countably infinite set Var of propositional variables and the following set of modalities:

(1) SA understood as 'A is settled true'; the dual modality is PA understood 'A is possible'.

(2) $[c]_aA$ understood as 'agent a *cstit*-realizes A'; the other action modality, namely, $[d]_aA$ to be read 'agent a *dstit*-realizes A', is in this setting a defined one with the following definition: $[c]_aA \wedge \neg SA$.

(3) I_aA understood as 'agent a imagines that A'.

Among other things, all the agent indices are assumed to stand for pairwise different agents.

For these modalities we assume the following 'stit-plus-neighborhood' semantics originally defined by H. Wansing in [2].

An imagination model is a tuple $\mathcal{M} = \langle Tree, \leq, Ag, Choice, \{N_a \mid a \in Ag\}, V\rangle$, where:

- $Tree$ is a non-empty set of moments, and $\leq$ is a partial order on $Tree$ such that

  $$\forall m_1, m_2 \exists m(m \leq m_1 \wedge m \leq m_2),$$

  and

  $$\forall m_1, m_2, m((m_1 \leq m \wedge m_2 \leq m) \to (m_1 \leq m_2 \vee m_2 \leq m_1)).$$

- The set $History$ of all histories of $\mathcal{M}$ is then just a set of all maximal $\leq$-chains in $Tree$. A history h is said to pass through a moment m iff $m \in h$. The set of all histories passing through $m \in Tree$ is denoted by H_m.

- Ag is a finite set of all agents acting in $Tree$ and is assumed to be disjoint from all the other items in $\mathcal{M}$.

- $Choice$ is a function defined on the set $Tree \times Ag$, such that for an arbitrary $\langle m, a\rangle \in Tree \times Ag$, we the value of this function, that is to say $Choice(m, a)$ (more commonly denoted $Choice^m_a$) is a partition of H_m. If $h \in H_m$, then $Choice^m_a(h)$ denotes the element of $Choice^m_a$, to which h belongs. In the special

case when we have $Choice^m_a = \{H_m\}$, it is said that the agent a has a *vacuous* choice at the moment m. In our models, $Choice$ is assumed to satisfy the following two restrictions:

- "No choice between undivided histories": for arbitrary $m \in Tree$, $a \in Ag$, $e \in Choice^m_a$, and $h, h' \in H_m$:

$$(h \in e \wedge \exists m'(m < m' \wedge m' \in h \cap h')) \to h' \in e.$$

- "Independence of agents". If f is a function defined on Ag such that $\forall a \in Ag(f(a) \in Choice^a_m)$, then $\bigcap_{a \in Ag} f(a) \neq \emptyset$.

- The set of moment-history pairs in $\mathcal{M}$, that is to say, the set

$$MH(\mathcal{M}) = \{\langle m, h\rangle \mid m \in Tree, h \in H_m\}$$

is then to be used as a set of points, where formulas are evaluated.

- For every $a \in Ag$, we have $N_a : MH(\mathcal{M}) \to 2^{(2^{MH(\mathcal{M})})}$. N_a is thus a neighborhood function, defining, for every moment history pair m/h the set of propositions imagined by the agent a at the moment m in history h.

- V is an evaluation function for atomic sentences, that is to say, $V : Var \to 2^{MH(\mathcal{M})}$.

The relation of satisfaction of sentences in the above defined language by moment-history pairs in $\mathcal{M}$ is then defined inductively as follows:

$$\mathcal{M}, m/h \vDash p \Leftrightarrow m/h \in V(p), \qquad \text{for atomic } p;$$
$$\mathcal{M}, m/h \vDash (A \wedge B) \Leftrightarrow \mathcal{M}, m/h \vDash A \wedge \mathcal{M}, m/h \vDash B;$$
$$\mathcal{M}, m/h \vDash \neg A \Leftrightarrow \mathcal{M}, m/h \nvDash A;$$
$$\mathcal{M}, m/h \vDash SA \Leftrightarrow \forall h' \in H_m(\mathcal{M}, m/h' \vDash A);$$
$$\mathcal{M}, m/h \vDash [c]_a A \Leftrightarrow \forall h' \in Choice^m_a(h)(\mathcal{M}, m/h' \vDash A);$$
$$\mathcal{M}, m/h \vDash I_a A \Leftrightarrow \forall h' \in Choice^m_a(h)(\{m/h \in MH(\mathcal{M}) \mid \mathcal{M}, m/h \vDash A\} \in N_a(m/h'))\wedge$$
$$\wedge\, \exists h'' \in H_m((\{m/h \in MH(\mathcal{M}) \mid \mathcal{M}, m/h \vDash A\} \notin N_a(m/h''))).$$

For this logic we propose the following axiomatization:
(A0) Propositional tautologies.
(A1) S is an $S5$ modality.
(A2) For every $a \in Ag$, $[c]_a$ is an $S5$ modality.
(A3) $SA \to [c]_a A$ for every $a \in Ag$.
(A4) $(P[c]_{a_1} A_1 \wedge \ldots \wedge P[c]_{a_n} A_n) \to P([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n)$, provided that all the $a_1 \ldots a_n$ are pairwise different.
(A5) $I_a A \to ([c]_a I_a A \wedge \neg S I_a A)$ for every $a \in Ag$.
Rules are as follows:
(R1) Modus ponens.
(R2) From A infer SA.
(R3) From $A \leftrightarrow B$ infer $I_a A \leftrightarrow I_a B$ for every $a \in Ag$.
Note. Thus the proposed axiomatization is just the axiomatization of *dstit* logic proposed by Ming Xu plus axiomatization of the logic of I_a as a minimal neighborhood

modal system **E** plus the special axiom (A5) stating the action character of the imagination operator. Note also that the converse of (A5) easily follows from (A2), so that we actually have a biconditional here.

Our aim now is to get a strong completeness theorem for this system L with respect to the above semantics, in the following form: if Θ is an L-consistent set of sentences, then Θ has a model in your proposed semantics.

In what follows we will always use 'consistency' to mean 'L-consistency' and we let $\vdash$ stand for a relation of L-derivability.

In order to get the main theorem, we use the technique of canonical models, which is an adaptation of the corresponding techniques for the two respective parts of our system as mentioned in the Note above. In particular, we draw on [1, ch. 17] in many matters relevant to the purely STIT part of the following construction.

More precisely, we let W to be the set of all L-maxiconsistent sets of sentences and we denote the members of W as w, w', w_1 etc. We set wRw' iff $\{A \mid SA \in w\} \subseteq w'$, and we set $w \simeq_a w'$ iff $\{A \mid [c]_a A \in w\} \subseteq w'$. By standard modal logic, (A1) and (A2) ensure that all these relations are relations of equivalence; moreover, (A3) ensures that $\simeq_a \subseteq R$ for every $a \in Ag$.

Indeed, let $w \simeq_a w'$ and let $SA \in w$. By (A3) and maxiconsistency of w, we get $[c]_a A \in w$, whence by $w \simeq_a w'$ we get that $A \in w'$. Since A was arbitrary, this means that wRw'.

In what follows, we will be denoting equivalence classes of W with respect to R by X, X', X_1, etc. The set of all such equivalence classes will be denoted by Ξ. When restricted to an arbitrary $X \in \Xi$, the relation R turns into a universal relation, but relations of the form $\simeq_a$ can remain non-trivial equivalences breaking X up into several equivalence classes. We will denote the family of equivalence classes corresponding to $\simeq_a \restriction X$ by $E(X, a)$.

Among the elements of W, we have a special interest in the maxiconsistent sets extending the following set of formulas:

$$\Sigma = \{\neg p \mid p \in Var\} \cup \{SA \leftrightarrow A \mid \text{ for arbitrary } A\} \cup \{[c]_a A \leftrightarrow A \mid \text{ for arbitrary } A\}.$$

The following facts are worth noting:

(F1) There exists exactly one element in W, which extends Σ. We will denote this element by $\mathbf{w}$. Indeed, one easily sees that Σ pre-determines every Boolean formula by fixing the literals. The modalities S and $[c]_a$ are then just vacuous in virtue of the definition of Σ. Finally, every maxiconsistent set extending Σ will have to contain $\neg I_a A$ for every formula A and every $a \in Ag$. For suppose otherwise. Then for some $w \in W$ such that $\Sigma \subseteq w$, for some formula A and for some $a \in Ag$ we will have $I_a A \in w$. Then, by (A5) and maxiconsistency of w we will get $\neg SI_a A \in w$. Therefore, by definition of Σ and maxiconsistency of w, we will get $\neg I_a A \in w$, which contradicts the assumption that $w \in W$. Therefore, the statements with I_a-modalities are also fixed for every $w \in W$, for which $\Sigma \subseteq w$. It is also easy to see that such a maxiconsistent w extending Σ must exist, since Σ itself is obviously consistent[1]

(F2) It follows from the definitions of Σ and R that the R-equivalence set containing $\mathbf{w}$, contains $\mathbf{w}$ only. We will denote this equivalence set by $\mathbf{X}$.

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[1] Σ is satisfiable and thus consistent. Indeed, consider a model consisting of a single moment, where every agent has a vacuous choice, every set of imagination neighborhoods is empty and every variable valuation is empty as well.

We now proceed to the definition of our canonical model. First, we choose[2] an element $0 \notin \Xi \cup W$ and define our set of moments:

$$Tree = \{0\} \cup \Xi \cup W.$$

We then set the following partial order on $Tree$. For arbitrary $x, y \in Tree$ we have $x \leq y$ iff $x = y$, or $y \in x$ or $x = 0$. This allows for a simple description of the set of histories in our frame. Every history turns out to have the form $h_w = \langle 0, X, w \rangle$, where $X \in \Xi$ and $w \in X$. Thus, our set of histories is in one-to-one correspondence with W.

Thirdly, we define the choice function. It assigns a vacuous choice to every agent at every moment m, if $m \notin \Xi$. That is to say, the only choice of every agent at every such moment will be just the set of all histories passing through this moment. Otherwise, i.e. for the case when $m = X \in \Xi$, we define the choice function as follows:

$$Choice^a_X = \{H \mid \exists e \in E(X, a)(H = \{h_w \mid w \in e\})\}.$$

Next, we need to define the imagination neighborhoods. We do this in the following way. $N_a(m/h) = \emptyset$ for every $a \in Ag$ and every $m \notin \Xi$. For the case when $m = X \in \Xi$, we need one further auxiliary notion. For every sentence A we set $Ext(A)$ (read: extension of A) to be $\{X/h_w \mid w \in X \wedge A \in w\}$ if $A \notin \mathbf{w}$; otherwise we set

$$Ext(A) = \{X/h_w \mid w \in X \wedge A \in w\} \cup \{m/h_w \mid m \notin \Xi \wedge m \in h_w\}.$$

Having defined the extensions, we set

$$N_a(X/h_w) = \{Ext(A) \mid I_a A \in w\}$$

for arbitrary $w \in X \in \Xi$.

Finally, we define the evaluation function for variables in the following way:

$$V(p) = \{X/h_w \mid w \in X \in \Xi \wedge p \in w\}.$$

We need to show that the canonical model $\mathcal{M}$ defined above is the model of our logic. The semantic restrictions are mostly seen to hold immediately; in particular, the no-choice-between-undivided-histories restriction holds because we only have undivided histories at the moment 0, where only vacuous choices are allowed. The only exception is the independence-of-agents restriction, which we treat below.

Lemma 1 (On Independence)**.** *Let $m \in Tree$ and let f be a function on Ag such that $\forall a \in Ag(f(a) \in Choice^a_m)$. Then $\bigcap_{a \in Ag} f(a) \neq \emptyset$.*

Proof. If $m \notin \Xi$, then the statement of the Lemma is obvious, since every agent will have a vacuous choice. We treat the case, when $m = X \in \Xi$. Consider a function f as described in Lemma. For every $f(a)$ we fix $e_{f(a)} \in E(X, a)$ such that $f(a) = \{h_w \mid w \in e_{f(a)}\}$ and we fix, further, an arbitrary $w_{f(a)} \in e_{f(a)}$. Since $e_{f(a)}$ is an $\simeq_a$-equivalence class, there is a set $\Gamma_{f(a)}$ of sentences of the form $[c]_a A$ shared by all the members of $e_{f(a)}$ and only those members. Also, since X is an R-equivalence

[2] We also assume, with the view of the definition of $\leq$ below, that 0 is not an element of any element of $\Xi \cup W$.

class, there is a set Δ of sentences of the form SA shared by all (and only) members of X. Consider, then, the following set of sentences:

$$\Lambda = (\bigcup_{a \in Ag} \Gamma_{f(a)}) \cup \Delta.$$

We claim that Λ is consistent. Assume otherwise. In this case Λ contains a finite inconsistent subset. Given that S and $[c]_a$ are $S5$-modalities, we can assume that this inconsistent subset has the following form:

$$SB, [c]_{a_1} A_1, \ldots, [c]_{a_n} A_n,$$

where all the $a_1 \ldots a_n$ are pairwise different (and moreover, $Ag = \{a_1 \ldots a_n\}$). We know, further, that for all $1 \leq i \leq n$ we have $SB, [c]_{a_i} A_i \in w_{f(a_i)}$. So, choose an arbitrary $w \in X$. For every $1 \leq i \leq n$ we have $w_{f(a_i)} R w$, therefore, we must also have $P[c]_{a_i} A_i \in w$ for every $1 \leq i \leq n$. Indeed, if it were otherwise, we would have $S\neg[c]_{a_i} A_i \in w$ since w is maxiconsistent. But then, given that $wRw_{f(a_i)}$, we would have $\neg[c]_{a_i} A_i \in w_{f(a_i)}$, a contradiction.

Thus, we have in fact that

$$P[c]_{a_1} A_1 \wedge \ldots \wedge P[c]_{a_n} A_n \in w,$$

therefore, by (A4), we also have

$$P([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n) \in w.$$

This, in turn, means that the set

$$\{A \mid SA \in \Delta\} \cup \{[c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n\}$$

is consistent: otherwise, we would have that

$$\{A \mid SA \in \Delta\} \vdash \neg([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n),$$

and, by standard modal $S5$-reasoning, that

$$\Delta \vdash S\neg([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n),$$

which, given that $w \in X$ and hence $\Delta \subseteq w$, would mean inconsistency of w, a contradiction.

Therefore, we may choose an arbitrary maxiconsistent w' extending $\{A \mid SA \in \Delta\} \cup \{[c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n\}$, and by the fact that this set contains $\{A \mid SA \in \Delta\}$ we know that wRw' and thus $w' \in X$ and further $SB \in w'$. This means that our finite subset in fact has a model and is not inconsistent. Therefore, since the finite set was arbitrary, Λ is consistent as well. Consider, then, an arbitrary maxiconsistent w'' extending Λ. Since $\Delta \subseteq w''$, we have $w'' \in X$, and since $\Gamma_{f(a)} \subseteq w''$ for arbitrary $a \in Ag$, we have $w'' \simeq_a w_{f(a)}$ for every such a. This means, in turn, that $w'' \in e_{f(a)}$ for every $a \in Ag$, and so $h_{w''} \in \bigcap_{a \in Ag} f(a) \neq \emptyset$. □

By now, the only ingredient to be added is the Truth Lemma; we divide it into two parts as follows.

Lemma 2 (Truth Lemma 1)**.** *Let $m \notin \Xi$ and $m \in h$. Then, for any sentence A, the following holds:*

$$\mathcal{M}, m/h \vDash A \Leftrightarrow A \in \mathbf{w}.$$

Proof. We use induction on the construction of A. If $A = p \in Var$, then $A \notin \mathbf{w}$, and also $m/h \notin V(A)$, since $m \notin \Xi$. Therefore, $\mathcal{M}, m/h \nvDash A$.

The boolean cases are then trivial.

If $A = SB$, then $\mathcal{M}, m/h \vDash A$ iff $\mathcal{M}, m/h' \vDash B$ for every h' such that $m \in h'$ iff $A \in \mathbf{w}$ by induction hypothesis (since we have proved IH for arbitrary h going through m).

If $A = [c]_a B$, then $\mathcal{M}, m/h \vDash A$ iff $\mathcal{M}, m/h' \vDash B$ for every h' such that $m \in h'$ and $h' \in Choice^a_m(h)$ iff $A \in \mathbf{w}$ by induction hypothesis (cf. the commentary on the previous case).

If $A = I_a B$, then $A \notin \mathbf{w}$ by (F1). We also have $\mathcal{M}, m/h \nvDash A$, since, given that $m \notin \Xi$, all the choices at m are vacuous. $\square$

Lemma 3 (Truth Lemma 2)**.** *Let $X \in \Xi$ and $w \in X$. Then, for any sentence A, the following holds:*

$$\mathcal{M}, X/h_w \vDash A \Leftrightarrow A \in w.$$

Proof. Again, we use induction on the construction of A. Atomic case we have by definition of V, and the boolean cases are obvious. We consider the modal cases.

Let $A = SB$, and assume that $SB \in w$. Then take any $h_{w'}$ passing through X. In the context of $\mathcal{M}$ this means that $w' \in X$, which in turn means that wRw'. Therefore, we have $B \in w'$ and, by induction hypothesis, $\mathcal{M}, X/h_{w'} \vDash B$. Since $h_{w'}$ was arbitrary, this means that $\mathcal{M}, X/h_w \vDash SB$.

On the other hand, assume that $SB \notin w$. This means that the set

$$\alpha = \{C \mid SC \in w\} \cup \{\neg B\}$$

is consistent. Indeed, otherwise we would have

$$\{C \mid SC \in w\} \vdash B,$$

and further, by standard $S5$ reasoning

$$\{SC \mid SC \in w\} \vdash SB,$$

and so, given, maxiconsistency of w, we would have $SB \in w$, contrary to our assumption. Therefore, consider an arbitrary $w' \in W$ extending α. By definition, $w' \in X$, therefore $h_{w'}$ goes through X and we have, by induction hypothesis, that $\mathcal{M}, X/h_{w'} \nvDash B$.

Let $A = [c]_a B$, and let $[c]_a B \in w$. Then take any $h_{w'}$ such that $h_{w'} \in Choice^a_X(h_w)$. In the context of $\mathcal{M}$ this means that $w \simeq_a w'$. Therefore, we have $B \in w'$ and, by induction hypothesis, $\mathcal{M}, X/h_{w'} \vDash B$. Since $h_{w'}$ was arbitrary, this means that $\mathcal{M}, X/h_w \vDash [c]_a B$.

On the other hand, assume that $[c]_a B \notin w$. This means that the set

$$\beta = \{C \mid [c]_a C \in w\} \cup \{\neg B\}$$

is consistent. Indeed, otherwise we would have

$$\{C \mid [c]_a C \in w\} \vdash B,$$

and further, by standard $S5$ reasoning

$$\{[c]_a C \mid [c]_a C \in w\} \vdash [c]_a B,$$

and so, given, maxiconsistency of w, we would have $[c]_a B \in w$, contrary to our assumption. Therefore, consider an arbitrary $w' \in W$ extending β. By definition, $w' \simeq_a w$, and also $w' \in X$ given that $\simeq_a \subseteq R$. Therefore $h_{w'}$ goes through X and moreover $h_{w'} \in Choice^a_X(h_w)$. By induction hypothesis, we have that $\mathcal{M}, X/h_{w'} \not\models B$, and so, putting all together, that $\mathcal{M}, X/h_w \not\models [c]_a B$

Let $A = I_a B$. First of all, note that by induction hypothesis and Lemma 2 we have the following biconditional:

$$Ext(B) = \{m/h \mid \mathcal{M}, m/h \models B\}. \tag{1}$$

Now, assume that $I_a B \in w$. Then, by (A5), we also have $[c]_a I_a B \in w$ and $\neg S I_a B \in w$. Take any $h_{w'}$ such that $h_{w'} \in Choice^a_X(h_w)$. In the context of $\mathcal{M}$ this means that $w \simeq_a w'$. Therefore, we have $I_a B \in w'$. By definition of N_a, this means that $Ext(B) \in N_a(X/h_{w'})$. On the other hand, the fact that $\neg S I_a B \in w$ means that the set

$$\gamma = \{C \mid SC \in w\} \cup \{\neg I_a B\}$$

is consistent. Indeed, otherwise we would have

$$\{C \mid SC \in w\} \vdash I_a B,$$

and further, by standard $S5$ reasoning

$$\{SC \mid SC \in w\} \vdash S I_a B,$$

and so, given, maxiconsistency of w, we would have $S I_a B \in w$, contrary to our assumption. Therefore, consider an arbitrary $w'' \in W$ extending γ. By definition, $w'' \in X$ so that $h_{w''}$ goes through X, and we have $Ext(B) \notin N_a(X/h_{w''})$ by definition of N_a.

Putting all this together, we get that, by (1), $\{m/h \mid \mathcal{M}, m/h \models B\} \in N_a(X/h_{w'})$ for every $h_{w'} \in Choice^a_X(h_w)$ and $\{m/h \mid \mathcal{M}, m/h \models B\} \notin N_a(X/h_{w''})$ for some $h_{w'}$ going through X. That is to say, we get that $\mathcal{M}, X/h_w \models I_a B$.

On the other hand, if $I_a B \notin w$, then, of course, $Ext(B) \notin N_a(X/h_w)$, and given the fact that $h_w \in Choice^a_X(h_w)$ and the biconditional (1), we get that $\mathcal{M}, X/h_w \not\models I_a B$ immediately. □

Now we are ready for our main result.

Theorem 1. *Let Θ be a consistent set of sentences. Then Θ has a model.*

Proof. Consider any maxiconsistent set w extending Θ and its corresponding R-equivalence class X. Then, by Lemma 3, we have $\mathcal{M}, X/h_w \models \Theta$. □

We also get **compactness** of L as a standard consequence of strong completeness.

References

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